Barker College

2003
YEAR 12
TRIAL HSC
EXAMINATION

MATHEMATICS

Staff Involved:

- PJR*
- RMH
- MRB
- BJR
- VAB
- LJP
- JML*
- GDH
- CFR
- GIC
- AJD

160 copies

DATE : AM Wednesday 6 August

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using blue or black pen
- Board-approved calculators may be used
- A Table of Standard Integrals is provided at the back of this paper
- ALL necessary working MUST be shown in every question

Total marks - 120

- Attempt Questions 1 - 10
- All questions are of equal value
- Answer EACH QUESTION on a NEW PIECE of lined paper
- Only write on ONE side of the lined paper
- Write your Barker Student Number at the top of each page of your answers

Total marks - 120
Attempt Questions 1 - 10
All questions are of equal value

Answer each question on a SEPARATE piece of lined paper.

Question 1 (12 marks) Use a NEW piece of lined paper. **Marks**

(a) Simplify $\dfrac{6x + 15x^3}{3x}$ **2**

(b) Solve $4x < 3(x - 1)$ **2**

Graph your solution on a number line.

(c) Find a primitive of $\sqrt{x^3} - 5$ **2**

(d) Express $\dfrac{\sqrt{6}}{\sqrt{6} - \sqrt{5}}$ in the form $a + b\sqrt{c}$, where a, b and c are integers. **2**

(e) Using the table of standard integrals, find $\int \sec 4x \tan 4x \; dx$ **1**

(f) Find the exact value of $\log_e e^2$ **1**

(g) The price of an article for sale is \$160, which includes GST of 10%. **2**

Calculate the price of the article without the GST.

Question 2 (12 marks) Start a NEW piece of lined paper. **Marks**

(a) Solve the pair of simultaneous equations **2**

$$y = 8 - 2x$$

$$x - y = 7$$

(b) The diagram below shows the parallelogram $OABC$ with vertices $O(0, 0)$, $A(3,5)$, $B(8,6)$ and C.

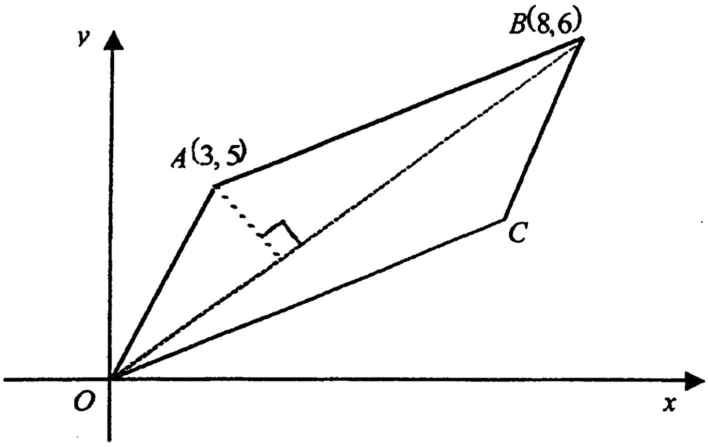


DIAGRAM NOT TO SCALE

(i) Write down the coordinates of the mid-point of OB. **1**

(ii) Find the coordinates of C. **1**

(iii) Show that the equation of the line OB is $3x - 4y = 0$ **2**

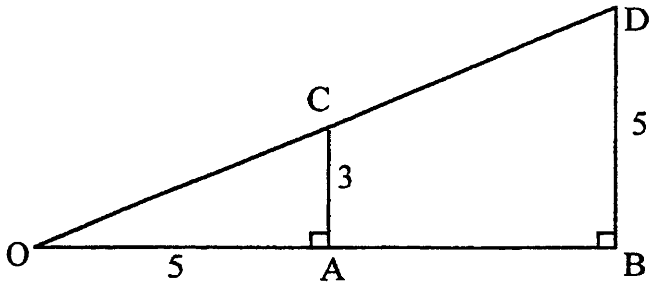
(iv) Show that the length of the interval OB is 10 units. **2**

(v) Calculate the perpendicular distance from A to the line OB. **2**

(vi) Calculate the area of the parallelogram $OABC$. **2**

Question 3 (12 marks) Start a NEW piece of lined paper. **Marks**

(a)



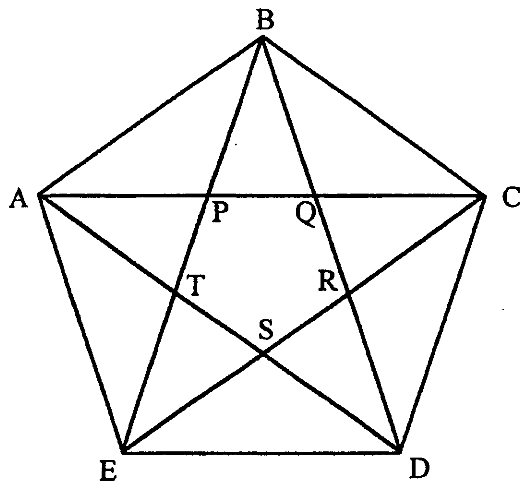
NOT TO SCALE

In the diagram above, AC is parallel to DB, OA = 5cm, DB = 5cm and AC = 3cm.

(i) Show that triangles OCA and ODB are similar. **3**

(ii) Hence, find the length of AB, giving reasons. **2**

(b)



ABCDE is a regular pentagon with each side being equal in length.
Equal diagonals have been drawn between all vertices to form another smaller regular pentagon PQRST.

(i) Find the size of $\angle CDE$. **1**

(ii) Show that triangles ADE and CDE are congruent. **3**

(iii) Find the size of $\angle DAE$, giving reasons . **1**

(iv) Hence, or otherwise, find the size of $\angle ATP$ giving reasons for your answer. **2**

Question 4 (12 marks) Start a NEW piece of lined paper. **Marks**

(a) Find the equation of the normal to the curve $y = x^3 - 3x + 1$ at the point (2, 3). **3**

(b) Differentiate with respect to x :

(i) $\ln(3x - 4)^5$ **2**

(ii) $3x^2 e^{x^3+1}$ **2**

(c) Find:

(i) $\int \left(x^2 + r^2\right) dx$ (where r is a constant) **2**

(ii) $\int_3^4 \frac{2}{x - 2}\, dx$ **3**

Question 5 (12 marks) Start a NEW piece of lined paper. **Marks**

(a) Consider the parabola with equation $(x + 3)^2 = 6(y - 1)$

(i) Find the coordinates of the vertex of the parabola **1**

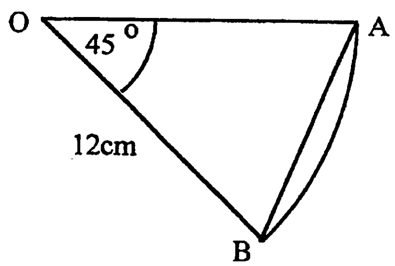
(ii) Find the coordinates of the focus of the parabola **2**

(b) For the quadratic equation $x^2 - (k - 1)x + (k - 2) = 0$, find the values of k for which the roots are real and different. **3**

(c) Find y such that $\log_e y + \log_e 3 = 2 \cdot 4$ **2**

(Give your answer as a decimal correct to two decimal places)

(d)



NOT TO SCALE

The diagram above shows a sector AOB of a circle with radius 12cm.

(i) Find the length of the arc AB. **2**

(ii) Find the length of the straight line AB. **2**

Question 6 (12 marks) Start a NEW piece of lined paper. **Marks**

(a) The first three partial sums of a series are $S_1 = 9$, $S_2 = 25$ and $S_3 = 50$ **2**

Find the first three terms, T_1, T_2 and T_3, of this series.

(b) The third term of a geometric series is 36 and the sixth term is 972

(i) Find the common ratio r **2**

(ii) Find the first term a **1**

(c) A super-bouncy ball is dropped from a height and takes 1 second to hit the ground.
The ball then bounces, taking a further $1\frac{1}{2}$ seconds before it bounces on the ground again.
Each successive bounce takes $\frac{3}{4}$ of the previous amount of time to bounce again.
Find how long the ball will be in motion before coming to rest. **3**

(d) William starts playing the game Space Cadet Pinball and gets an initial high score of 858.
He then regularly plays this game and keeps a record of the <u>improvement</u> in his high scores at the end of each week. These are recorded in the table below.

Week	1	2	3	4	5	6	...	14	15	...
Improvement	298	281	264	247	230	213	...	77	60	...

(i) How many weeks will it take for William to reach his overall maximum score? **2**

(ii) What will be William's overall maximum score? **2**

Question 7 (12 marks) Start a NEW piece of lined paper. **Marks**

(a) A resistance force, F, is related to the speed, v, by the function

$$F(v) = v^2 + \frac{16}{v}$$

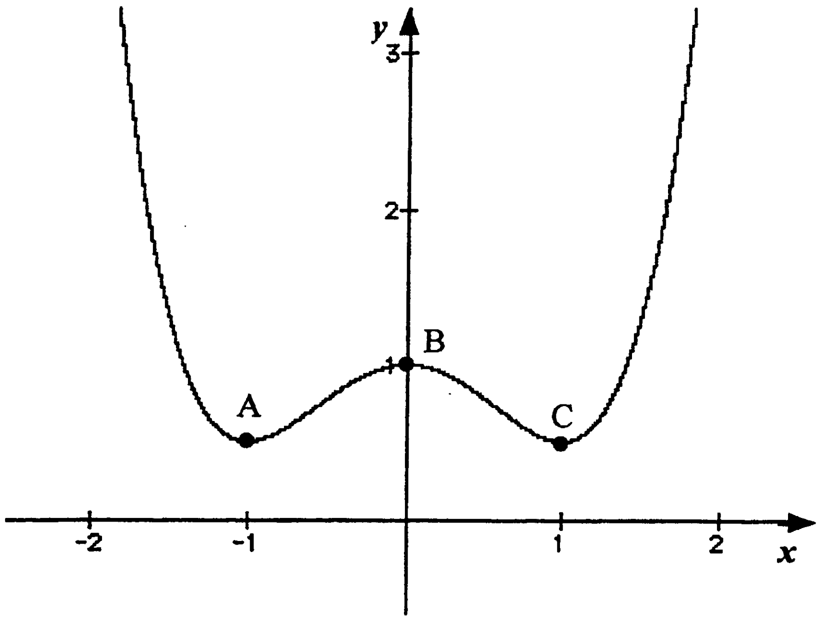
(i) Find $F'(v)$ **2**

(ii) This resistance force is minimised when $F'(v) = 0$ **2**

Find the speed at which this resistance force is minimised.

(b) The graph of $y = \frac{1}{2}x^4 - x^2 + 1$ is sketched below.

The points A, B and C are the stationary points of this curve.



(i) Find the coordinates of the points A, B and C. **3**

(ii) For what values of x is this curve concave down? **3**

Give reasons for your answer.

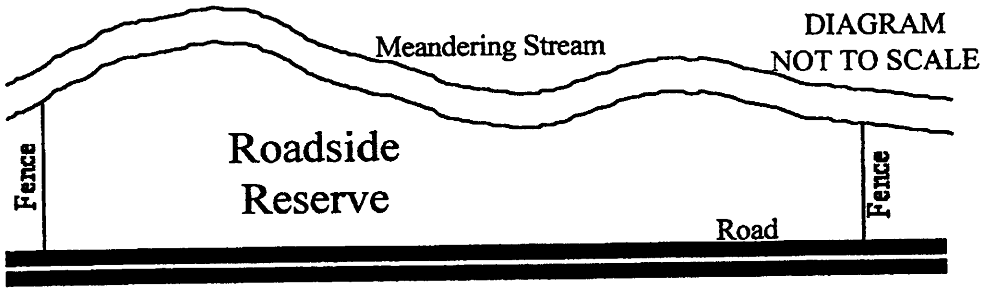
(iii) Using Part (i), draw a rough sketch of the gradient function, $\frac{dy}{dx}$, of this curve. **2**

Identify any points on this sketch where the concavity of the curve changes.

Question 8 (12 marks) Start a NEW piece of lined paper. **Marks**

(a) Find the equation of the curve that passes through the point (5, 4) and has a gradient function of $2x - 6$ **2**

(b) A roadside reserve is bounded by a straight road, a meandering stream and two straight fences to neighbouring farms. The border along the road is 120 metres.



The width of the roadside reserve from the road-edge to the stream-bank, measured at 20 metre intervals along the road is given in the table below.

Distance along road (m)	0	20	40	60	80	100	120
Width of roadside reserve (m)	20	30	22	16	18	18	14

Use Simpson's Rule to approximate the area of this roadside reserve. **3**

(c) (i) On the same axes sketch $y = x^2 - x$ and $y = x + 3$ **2**

(ii) Find the x values of the points of intersection of these two curves. **2**

(iii) Hence, find the area between these two curves. **3**

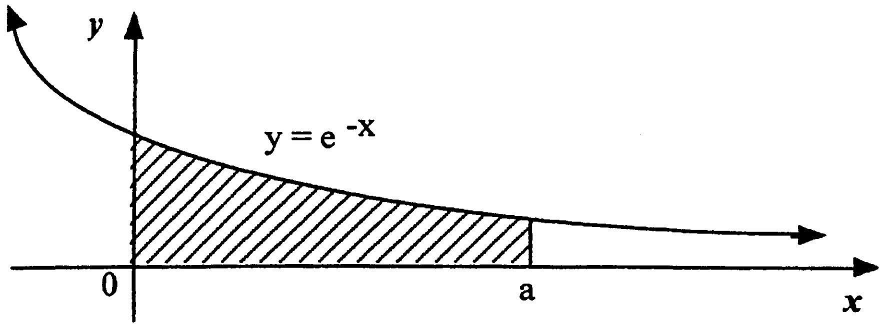
Question 9 (12 marks) Start a NEW piece of lined paper. **Marks**

(a) (i) Without using calculus, sketch the curve $y = e^x - 2$ **2**

(ii) On the same sketch, find, graphically, the number of solutions of the equation **2**

$$e^x - x - 2 = 0$$

(b)



(i) What is the volume of the solid formed when the shaded area is rotated completely around the x-axis? **2**

(ii) What is the limit of this volume as $a \to \infty$? **1**

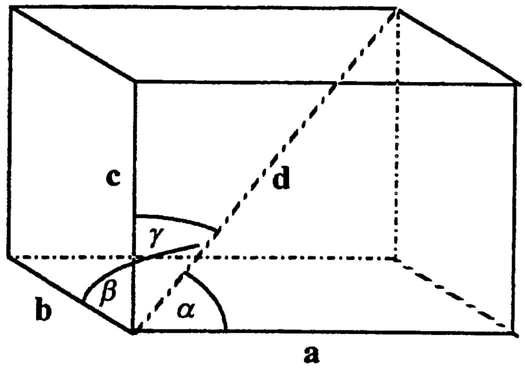
(c) Amelia has borrowed \$5000 at the beginning of 2003.
The debt is to be repaid by equal annual installments of \$1200.
The first installment is to be repaid at the beginning of 2004.
Interest at the rate of 18% p.a. is calculated at the beginning of each year on the balance owing at the end of the previous year.
This interest is then added to the balance of the debt before a repayment is made.

(i) Calculate the amount owing on the debt after the first repayment is made at the beginning of 2004. **2**

(ii) Amelia wants to clear her debt at the beginning of 2010. **3**
How much extra will she have to repay at the beginning of 2010 (after making her normal repayment) in order for her to do this?

Question 10 (12 marks) Start a NEW piece of lined paper. **Marks**

(a)



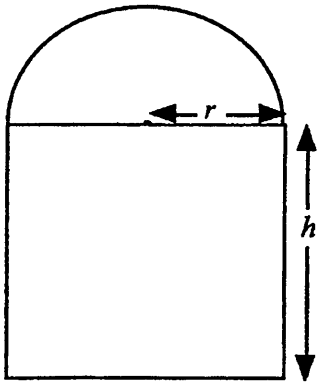
A rectangular box has edges of length ***a***, ***b*** and ***c***. A diagonal of length ***d*** is drawn through the box between opposite corners as shown.
The three different angles between this diagonal and the three edges ***a***, ***b*** and ***c*** of the box are labeled α, β and γ respectively.

(i) Express ***d*** in terms of ***a***, ***b*** and ***c***. **2**

(ii) Hence, or otherwise, show that the angles α, β and γ obey the identity **3**

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

(b) A window in the chapel has been damaged by a storm and needs to be replaced.



It is in the shape of a rectangle surmounted by a semi-circle, as shown.

Let the radius of the semi-circle be *r* metres and the height of the rectangle be *h* metres.

(i) Given that the perimeter of the window is to be 10π metres, show that **2**

$$h = 5\pi - r - \frac{\pi r}{2}$$

(ii) Hence, show that the area of the window is given by the formula **1**

$$A = 10\pi r - 2r^2 - \frac{1}{2}\pi r^2$$

(ii) Hence, find the radius of the circle for which the area of the window is to be a maximum. **4**

END OF PAPER

2003 2 Unit Mathematics Trial HSC Solutions

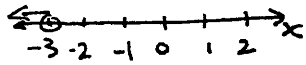
Question 1

(a) $\frac{3x(2+5x^2)}{3x}$

$= 2+5x^2$

(b) $4x < 3x-3$

$x < -3$



(c) $\int x^{3/2} - 5\, dx$

$= \frac{2x^{5/2}}{5} - 5x + C$

(d) $\frac{\sqrt{6}}{(\sqrt{6}-\sqrt{5})} \times \frac{(\sqrt{6}+\sqrt{5})}{(\sqrt{6}+\sqrt{5})}$

$= \frac{6+\sqrt{30}}{6-5}$

$= 6+\sqrt{30}$

(e) $\frac{1}{4}\sec 4x + C$

(f) $2\log_e e$

$= 2$

(g) $110\% = \$160$

$10\% = \frac{160}{11}$

$\therefore 100\% = \frac{160}{11} \times 10$

$= \$145.45$

Question 2

(a) $2x+y=8$, $x-y=7$ $\therefore 3x = 15$, $x = 5$

$\therefore 10+y=8$, $y=-2$

$x=5, y=-2$

(b)(i) Mid-pt of OB $= \left(\frac{8}{2}, \frac{6}{2}\right)$

$= (4,3)$

(ii) $C \Rightarrow (8-3, 6-5)$

C is $(5,1)$

(iii) $m = \frac{6-0}{8-0} = \frac{3}{4}$

Eqn is

$y-0 = \frac{3}{4}(x-0)$

$\therefore 4y = 3x$

$\therefore 3x - 4y = 0$

(iv) $d_{OB} = \sqrt{(8-0)^2 + (6-0)^2}$

$= \sqrt{64+36}$

$= \sqrt{100} = 10$ units

(v) $(3,5)$ $3x-4y=0$

$d = \frac{|3\times 3 + 5\times -4 + 0|}{\sqrt{3^2+(-4)^2}}$

$= \frac{|9-20|}{\sqrt{9+16}}$

$= \frac{|-11|}{\sqrt{25}} = \frac{11}{5}$ units

(vi) Area of ||ogram $= 2 \times$ Area of ΔOAB

$= 2\times\left(\frac{1}{2}\times 10 \times \frac{11}{5}\right)$

$= 22$ units^2

Question 3

(a)(i) $O\hat{A}C = O\hat{B}D = 90°$ (right angles given)

$C\hat{O}A = D\hat{O}B$ (common angle)

$O\hat{C}A = O\hat{D}B$ (corres. ∠s on || lines are equal)

$\therefore \Delta OCA ||| \Delta ODB$ (equal angles)

(ii) $\frac{AC}{BD} = \frac{OA}{OB}$ (sides of similar Δs in same ratio)

$\therefore \frac{3}{5} = \frac{5}{5+AB}$

$\therefore 15 + 3AB = 25$

$\therefore 3AB = 10$

$\therefore AB = \frac{10}{3}$ cm

(b)(i) Angle sum of ABCDE $= (5-2)\times 180$

$= 3 \times 180$

$= 540°$

$\therefore C\hat{D}E = \frac{540}{5}$

$= 108°$

(ii) ED is common

AE = CD (equal sides of regular pentagon)

$C\hat{D}E = A\hat{E}D$ (equal angles of regular pentagon)

$\therefore \Delta ADE \equiv \Delta CDE$ (SAS)

(iii) ΔADE is isos. Δ (AE = DE equal sides of pentagon)

thus $D\hat{A}E = A\hat{D}E$

$= \left(\frac{180-108}{2}\right)$ (Equal base ∠s of isos Δ)

$= 36°$

(iv) $P\hat{T}S = 108°$ (equal angles of regular pentagon)

$\therefore A\hat{T}P = 180 - 108$ (straight line angle)

$= 72°$

Question 4

(a) $y = x^3 - 3x + 1$

$\frac{dy}{dx} = 3x^2 - 3$

When $x = 2$, $y' = 3\times 2^2 - 3$

$\therefore m = 9$

$\therefore$ Grad of normal $= -\frac{1}{9}$

Eqn of normal is

$y - 3 = -\frac{1}{9}(x-2)$

$\therefore 9y - 27 = -x + 2$

$\therefore x + 9y - 29 = 0$

(b)(i) $y = \ln(3x-4)^5$

$y = 5\ln(3x-4)$

$\therefore y' = 5 \times \frac{3}{3x-4}$

$= \frac{15}{3x-4}$

(ii) $y = 3x^2 e^{x^3+1}$

$u = 3x^2$ $u' = 6x$

$v = e^{x^3+1}$ $v' = 3x^2 e^{x^3+1}$

$\frac{dy}{dx} = 3x^2 \times 3x^2 e^{x^3+1} + 6x \times e^{x^3+1}$

$= 9x^4 e^{x^3+1} + 6xe^{x^3+1}$

(c)(i) $\int x^2 + r^2\, dx$

$= \frac{x^3}{3} + r^2 x + C$

(ii) $2\int_3^4 \frac{1}{x-2}\, dx$

$= 2\left[\ln(x-2)\right]_3^4$

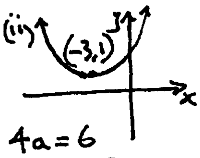
$= 2\{\ln 2 - \ln 1\}$

$= 2\ln 2$

Question 5

(a)(i) Vertex $\Rightarrow (-3, 1)$

(ii)



$4a = 6$

$\therefore a = \frac{3}{2}$

$\therefore$ Focus $\Rightarrow (-3, 2\frac{1}{2})$

(b) $\Delta > 0$ if roots real and different

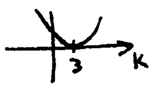
$\Delta = (k-1)^2 - 4\times 1 \times (k-2)$

$= k^2 - 2k + 1 - 4k + 8$

$= k^2 - 6k + 9$

$\therefore k^2 - 6k + 9 > 0$

$\therefore (k-3)^2 > 0$



$\therefore k < 3$ or $k > 3$

(c) $\log_e 3y = 2.4$

$\therefore 3y = e^{2.4}$

$\therefore y = \frac{e^{2.4}}{3}$

$\doteqdot 3.67$

(d)(i) $45° = \frac{\pi}{4}$ radians

$l = r\theta \Rightarrow l = 12 \times \frac{\pi}{4}$

$\therefore l = 3\pi$ cm

$\doteqdot 9.42$ cm

(ii) $OA = OB = 12$

$c^2 = 12^2 + 12^2 - 2\times 12 \times 12 \times \cos 45$

$c^2 = 84.3532\ldots$

$\therefore c \doteqdot 9.18$ cm

Question 6

(a) $S_1 = T_1 = 9$

$S_2 = T_1 + T_2$

$\therefore 25 = 9 + T_2$

$\therefore T_2 = 16$

$S_3 = T_1 + T_2 + T_3$

$\therefore 50 = 9 + 16 + T_3$

$\therefore T_3 = 25$

(b) $T_3=36, T_6=972$

(i) $ar^5=972$, $ar^2=36$

$\frac{ar^5}{ar^2}=\frac{972}{36}$

$\therefore r^3=27$

$\therefore r=3$

(ii) $a\times 3^2=36$

$\therefore a=\frac{36}{9}$

$\therefore a=4$

(c) $1+\frac{3}{2}+\frac{3}{2}\times\frac{3}{4}+\frac{3}{2}\times\left(\frac{3}{4}\right)^2$

$=1+$ Sum of infinite G.P. with $a=\frac{3}{2}, r=\frac{3}{4}$

$=1+\frac{\frac{3}{2}}{1-\frac{3}{4}}$

$=1+6$

$=7$ seconds comes to rest

(d)(i) $a=298, d=-17$

Overall max score when $T_n=0$

$0=298+(n-1)\times -17$

$0=298-17n+17$

$17n=315$

$n=18.5$ ie. in 19 weeks

(ii) $T_{18}=298+17\times -17$

$=9$

$\therefore$ Max Score $=858+S_{18}+9$

$S_{18}=\frac{18}{2}[2\times 298+17\times -17]$

$=9(596-289)$

$S_{18}=2763$

$\therefore$ Max Score $=3630$

Question 7

(a)(i) $F(v)=v^2+16v^{-1}$

$F'(v)=2v-16v^{-2}$

(ii) $0=2v-\frac{16}{v^2}$

$\therefore \frac{16}{v^2}=2v$

$\therefore 16=2v^3$

$\therefore v^3=8$

$\therefore v=2$ when force is minimised

(b)(i) $y=\frac{1}{2}x^4-x^2+1$

$\frac{dy}{dx}=2x^3-2x$

Stat pts $\Rightarrow \frac{dy}{dx}=0$

$\therefore 2x^3-2x=0$

$\therefore 2x(x^2-1)=0$

$\therefore 2x(x+1)(x-1)=0$

$\therefore x=0,-1,1$

When $x=0, y=1$ B is $(0,1)$

When $x=\pm 1, y=\frac{1}{2}-1+1$

$=\frac{1}{2}$

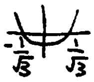
$\therefore A=(-1,\frac{1}{2})$ $C=(1,\frac{1}{2})$

(ii) $y''=6x^2-2$

Concave down $\Rightarrow y''<0$

$\therefore 6x^2-2<0$

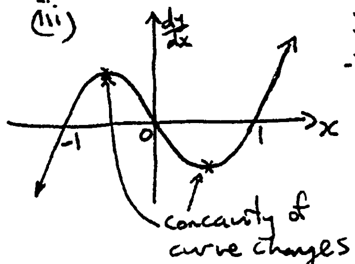
$x^2-\frac{1}{3}<0$



$(x-\frac{1}{\sqrt{3}})(x+\frac{1}{\sqrt{3}})<0$

$\therefore -\frac{1}{\sqrt{3}}<x<\frac{1}{\sqrt{3}}$

(iii)



Question 8

(a) $\frac{dy}{dx}=2x-6$

$y=x^2-6x+C$

Subst. $(5,4)\Rightarrow 4=25-30+C$

$\therefore 4=-5+C$

$c=9$

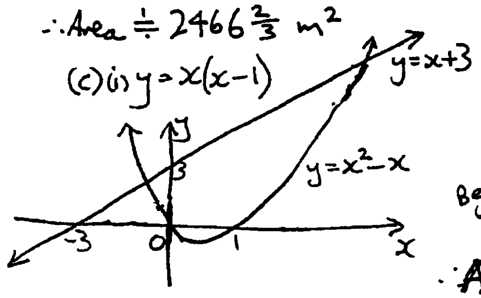
curve is $y=x^2-6x+9$

(b) $h=20$

Area $\doteqdot \frac{20}{3}[(20+14)+4\times(30+16+18)+2\times(22+18)]$

$\therefore$ Area $\doteqdot 2466\frac{2}{3}$ m^2

(c)(i) $y=x(x-1)$



(ii) $y=x^2-x$, $y=x+3$

$\therefore x^2-x=x+3$

$\therefore x^2-2x-3=0$

$\therefore (x-3)(x+1)=0$

$\therefore x=3$ or -1

(iii) $A=\int_{-1}^{3}(x+3)-(x^2-x)\,dx$

$=\int_{-1}^{3}3+2x-x^2\,dx$

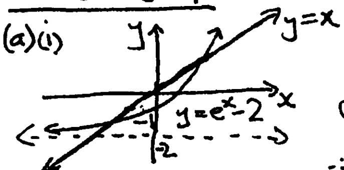
$=\left[3x+x^2-\frac{x^3}{3}\right]_{-1}^{3}$

$=(9+9-\frac{27}{3})-(-3+1+\frac{1}{3})$

$=10\frac{2}{3}$ units^2

Question 9

(a)(i)



(ii) $e^x-2=x$

$y=e^x-2$ and $y=x$

$\therefore$ 2 solutions to eqn

(b)(i) $V=\pi\int_0^a (e^{-x})^2\,dx$

$=\pi\int_0^a e^{-2x}\,dx$

$=\pi\left[\frac{e^{-2x}}{-2}\right]_0^a$

$=\frac{\pi}{-2}(e^{-2a}-e^0)$

$=-\frac{\pi}{2}(e^{-2a}-1)$ units^3

(ii) $V=-\frac{\pi}{2}\left(\frac{1}{e^{2a}}-1\right)$

As $a\to\infty$, $\frac{1}{e^{2a}}\to 0$

$\therefore V\to\frac{\pi}{2}$ units^3

(c)(i) $A_1=5000\times 1.18-1200$

$=\$4700$

(ii) $A_2=A_1\times 1.18-1200$

$=5000\times 1.18^2-1200(1.18+1)$

$A_3=A_2\times 1.18-1200$

$=5000\times 1.18^3-1200(1.18^2+1.18+1)$

Begin of 2010 $\Rightarrow A_7=5000\times 1.18^7-1200(1.18^6+\ldots+1.18+1)$

G.P. with $a=1, r=1.18, n=7$

$\therefore A_7=5000\times 1.18^7-1200\times\frac{1\times(1.18^7-1)}{1.18-1}=\$1357.54$

Question 10

(a)(i) Diagonal of base $=\sqrt{a^2+b^2}$

$\therefore d^2=c^2+\left(\sqrt{a^2+b^2}\right)^2$

$\therefore d^2=a^2+b^2+c^2$

$\therefore d=\sqrt{a^2+b^2+c^2}$

(ii) $\cos\alpha=\frac{a}{d}$, $\cos\beta=\frac{b}{d}$ and $\cos\gamma=\frac{c}{d}$

LHS $=\cos^2\alpha+\cos^2\beta+\cos^2\gamma$

$=\frac{a^2}{d^2}+\frac{b^2}{d^2}+\frac{c^2}{d^2}$

$=\frac{a^2+b^2+c^2}{d^2}$

$=\frac{d^2}{d^2}=1=$ RHS

(b)(i) $10\pi=2h+2r+\pi r$

$\therefore 2h=10\pi-2r-\pi r$

$\therefore h=5\pi-r-\frac{\pi r}{2}$

(ii) $A=2rh+\frac{1}{2}\pi r^2$

$\therefore A=2r\left(5\pi-r-\frac{\pi r}{2}\right)+\frac{\pi r^2}{2}$

$\therefore A=10\pi r-2r^2-\pi r^2+\frac{\pi r^2}{2}$

$\therefore A=10\pi r-2r^2-\frac{\pi r^2}{2}$

(iii) $A'=10\pi-4r-\pi r$

Max when $A'=0\Rightarrow$

$\therefore 0=10\pi-4r-\pi r$

$\therefore \pi r+4r=10\pi$

$\therefore r(\pi+4)=10\pi$

$\therefore r=\frac{10\pi}{\pi+4}$ $(\doteqdot 4.4)$

r	4	4.4	5
A'	2.8	0	-4.3
	/	—	\

Thus, maximum when $r=\frac{10\pi}{\pi+4}$ metres